Time Sub-optimal Nonlinear PI and PID Controllers applied to Longitudinal Headway Car Control

Minh-Duc Hua\textsuperscript{a} and Claude Samson\textsuperscript{b}

\textsuperscript{a}I3S UNS–CRNS, Nice-Sophia Antipolis, France
email: minh.hua@polytechnique.org

\textsuperscript{b}INRIA Sophia Antipolis–Méditerranée, Sophia-Antipolis, France
email: Claude.Samson@inria.fr

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Nonlinear PI and PID controllers combining time-(sub)optimality with linear control robustness and anti-windup properties are proposed for first-order and second-order integrator systems, without assuming that the control lower-bound and upper-bound are the opposite of each other. A complementary contribution is the introduction of an integral action with anti-windup properties into the control law, under the constraint of ensuring global asymptotic stability. For illustration purposes, the proposed PID solution is applied to the longitudinal headway control of a vehicle following another vehicle.

Keywords: time (sub)-optimal control; nonlinear PI/PID; anti-windup; conditional integrator; nested saturation; longitudinal headway car control

1 Introduction

Proportional-integral (PI) and proportional-integral-derivative (PID) controllers are at the heart of control engineering practice and, owing to their relative simplicity and satisfactory performance for a wide range of processes, have become the standard controllers used by industry. Following an estimation of Koivo and Tanttu (1991), perhaps only 5-10\% of man-implemented control loops cannot be controlled by single-input single-output (SISO) PI or PID controllers. However, this widespread usage also goes with numerous problems due to either poor tuning practice or limited capabilities offered by standard PI-PID schemes. These problems have in turn periodically revived the interest from the academic research community in order to work out complementary explanations and solutions Aström and Hågglund (1995), Hippe (2006), O’Dwyer (2009). In particular, a well-known source of degradation of performance is the occurrence of control saturation, when the boundedness of the “physical” control that can be applied to the system under consideration is no longer compatible with the application of the (theoretically unbounded) calculated control value. This has the consequence of invalidating the performance index established on the assumption of linearity of the controlled system, and can give rise to various undesired (and unnecessary) effects such as multiple bouncing between minimal and maximal values of the control, and important overshoots of the regulated error variables. The so-called integrator windup phenomenon, which worsens the overshoot problem and the reduction of which still motivates various research studies Aström and Hågglund (1995), Aström and Rundqvist (1989), Hippe (2006), Hodel and Hall (2001), Peng et al. (1996), Seshagiri and Khalil (2005), is also commonly presented as a consequence of control saturation combined with the
integral action incorporated in the control law in order to compensate for unknown (slowly varying) additive perturbations.

Compared to the already huge corpus of studies devoted to PI and PID controllers, the present paper has the limited ambition of proposing new nonlinear versions of these controllers that attempt to combine the constraints of control saturation with

\begin{enumerate}
\item the objective of optimizing the control action in order to reduce the size of initially large tracking errors as fast as possible, and
\item the design of integral action terms with limited windup effects:
\end{enumerate}

\begin{itemize}
\item The former issue is close to the line of research on “proximate” time-optimal for linear systems admitting closed-form time-optimal solutions Newman (1990), Pao and Franklin (1993), Workman et al. (1987). The present study is restricted to the simplest first and second order linear systems. In particular, continuous nonlinear proportional (P) and proportional-derivative (PD) state feedbacks, which depend continuously on an extra-parameter whose convergence to infinity yields the discontinuous time-optimal controls for these systems, are derived and form the cores of the nonlinear PI and PID controllers proposed subsequently. Whereas most works on “proximate” time-optimal control assume that the control lower-bound and upper-bound are the opposite of each other Newman (1990), Pao and Franklin (1993), Workman et al. (1987), this restrictive assumption is not made in the present paper. The removal of this assumption complicates the control design and involves non-standard stability and convergence analyses which constitute another original contribution of this work.
\item As for the latter issue, it is related to the work on anti-windup and “conditional integrators” as exemplified in Kothare et al. (1994), Seshagiri and Khalil (2005). For instance, the conditional integrator proposed in Seshagiri and Khalil (2005) is adapted to our controller in the case of the first-order system. A novel anti-windup integrator is proposed for the second-order system.
\end{itemize}

The present work is also related to the theme of bounded control design based on the use of nested saturation functions Teel (1992), Marconi and Isidori (2000), Wang and Ma (2010), with the same concern of proving \textit{global} asymptotic stability of the desired set-point, but with a different way of designing the control solutions.

In the second part of the paper, the proposed nonlinear PID controller is applied to the longitudinal headway control of a car following a leader. The reason for choosing this application is its good fit with the design constraints and objectives imposed on the control and its performance; namely

\begin{enumerate}
\item the existence of different bounds on the car’s acceleration and deceleration capabilities,
\item control effectiveness in terms of time of convergence to the desired inter-distance between the two vehicles,
\item absence of bouncing transients –for the comfort of the passengers, fuel economy, and reduced wear-off of mechanical parts– and
\item very small overshoot in order to avoid collisions with the leader. This type of application has also motivated numerous studies in the last two decades, see for instance Hatipoğlu et al. (1996), Zhang et al. (1999), Martinez and de Wit (2007) among many other contributions. The results here presented are based on a simple model of the system’s dynamics which would obviously call for several refinements of practical relevance, and they by no means aim at covering the subject in depth. The purpose is just to point out a novel and simple PID solution which basically addresses the same issues as in Hatipoğlu et al. (1996), with the economy of a switching strategy, and that experts on the subject might consider in the future.
\end{enumerate}

A preliminary version of this work has been accepted for a conference Hua and Samson (2011). The present paper extends this version by completing the results in Lemmas 2.1 and 3.2 with difficult technical results and their proofs –here reported in the Appendix– that have been worked out later on.
2 Recalls on time-optimal controls (TOC) for first-order and second-order integrators, and continuous feedback approximations

2.1 First-order system

Let $M > 0$ and $m < 0$ denote two real numbers. In what follows $sat^M_m$ denotes the saturation function defined on $\mathbb{R}$ by

$$
sat^M_m(x) = \begin{cases} 
M & x \geq M \\
m & x \leq m \\
 x & \text{if } x \in [m, M]
\end{cases}
$$

(1)

To simplify the notation, we will write $sat^M(x)$ instead of $sat^M_M(x)$ when $m + M = 0$. The results described in this paper could in fact be adapted to a more general class of saturation functions, including those based on the use of $tanh$. Define also the discontinuous $sign^M_m$ function as follows

$$
sign^M_m(x) = \begin{cases} 
M & x > 0 \\
0 & x = 0 \\
m & x < 0
\end{cases}
$$

(2)

One remarks that $sign^M_m(\cdot) = \lim_{k \to +\infty} sat^M_m(k \cdot)$. Consider the first order integrator

$$
\dot{x} = u,
$$

(3)

with the control variable $u$ such that $m \leq u \leq M$. The TOC associated with this system that takes $x$ to zero in minimal time can be written as

$$
u(x) = sign^M_m(-x).
$$

Due to the discontinuity at the desired set-point, this is not a good feedback law. Indeed, even though it theoretically stabilizes $x = 0$ asymptotically (when considering solutions of the controlled system defined in the sense of Filippov, for instance), it is excessively sensitive to measurement noise and chattering, and its discretization systematically renders the origin unstable. A continuous approximation of this optimal control, endowed with better robustness properties around the origin, is given by

$$
u(x) = sat^M_m(-k_p x), \quad k_p > 0,
$$

(4)

with the approximation improving uniformly (in terms of reaching a given small neighborhood of $x = 0$ from any initial condition) by increasing the value of $k_p$. Locally, near the origin, this latter control is equal to the proportional feedback law

$$
u(x) = -k_p x.
$$

It thus locally inherits the properties of this linear feedback control, whereas it approximates the TOC when the “error” $x$ is initially large. In practice, the gain $k_p$ can be tuned according to Linear Control Theory rules, typically in relation to control sampling, measurement noise, and additive perturbation issues addressed by control performance and robustness analyses.
2.2 Second-order system

Consider now the second-order integrator

\[ \ddot{x} = u \tag{5} \]

with the same bound constraints as previously, i.e. \( m \leq u \leq M \). Set

\[ s_a(x, \dot{x}) = \begin{cases} 
\frac{x + \dot{x}|\dot{x}|}{2a} & x + \frac{\dot{x}|\dot{x}|}{2a} \neq 0 \\
\dot{x} & \text{if } x + \frac{\dot{x}|\dot{x}|}{2a} = 0, \quad x^2 + \dot{x}^2 \neq 0 \\
0 & x + \frac{\dot{x}|\dot{x}|}{2a} = 0, \quad x^2 + \dot{x}^2 = 0 
\end{cases} \]

The TOC associated with this system that takes \( x \) to zero in minimal time can be written as

(see, e.g., Athans and Falb (1966), Kirk (1970))

\[ u(x, \dot{x}) = \text{sign}_m \left( -s_a(x, \dot{x}) \right), \quad a = \begin{cases} 
M & \text{if } x \geq 0 \\
-m & \text{if } x < 0 
\end{cases} \]

a simplification of which is

\[ u(x, \dot{x}) = \text{sign}_m \left( -\left( x + \frac{\dot{x}|\dot{x}|}{2a} \right) \right), \quad a = \begin{cases} 
M & \text{if } x \geq 0 \\
-m & \text{if } x < 0 
\end{cases} \]

This feedback law is discontinuous at points \((x, \dot{x})\) where \( x + \frac{\dot{x}|\dot{x}|}{2a} = 0 \) \((a = M, -m)\), and also on the line \( x = 0 \). It is in particular discontinuous at the desired equilibrium \((x, \dot{x}) = (0, 0)\). In order to ensure continuity at this point one may consider the following approximation:

\[ u(x, \dot{x}) = \text{sat}_m^M \left( -k_p \left( x + \frac{\dot{x}|\dot{x}|}{2a(x, \varepsilon)} \right) \right) - \text{sat}_l^l \left( k_v \dot{x} \right), \tag{6} \]

with \( l > 0 \) bounding the interval on which this linear term is the most active, and \( k_v > 0 \) playing the role of a “derivative gain”. Indeed, the linear approximation of the above feedback at the
desired equilibrium \((x = 0, \dot{x} = 0)\) is the classical PD controller

\[
u(x, \dot{x}) = -k_p x - k_v \dot{x},
\]

whose proportional and derivative gains, \(k_p\) and \(k_v\), can be determined by applying classical rules of Linear Control Theory. For instance, for the double integrator system \(\ddot{x} = u\), the choice \(k_v = 2\sqrt{k_p}\) yields two closed-loop poles equal to \(-\sqrt{k_p}\) and ensures a critically-damped response with no overshoot. As for the choice of the parameter \(l\), it corresponds to a compromise between (local) robustness (as provided by a linear PD feedback) and performance when starting far away from the desired equilibrium (as provided by the time-optimal nonlinear control). In Newman (1990), the author studied the particular case where \(M + m = 0\) and, by considering a sliding-mode formulation, derived a nonlinear PD controller which is essentially the same as (8).

### 2.3 Stability and convergence

Let us analyze the stability and convergence properties associated with the time sub-optimal controller (4) (resp. (8)) applied to the first-order integrator system (3) (resp. second-order integrator system (5)). From the fact that the linear approximations of these controllers coincide with classical P and PD feedbacks one can already deduce that they are local exponential stabilizers. The following lemma points out that they are in fact global asymptotic stabilizers.

**Lemma 2.1:**

1. The nonlinear proportional feedback control (4) globally asymptotically stabilizes \(x = 0\) for the first-order integrator system (3).
2. The nonlinear proportional–derivative feedback control (8), with \(a(x, \varepsilon)\) chosen either positive constant or according to relation (7) with \(0 < \varepsilon < \min(-m, M)/k_p\), globally asymptotically stabilizes \((x, \dot{x}) = (0, 0)\) for the second-order integrator system (5).

**Proof** For the P controller (4), and the PD controller (8) with \(a(\cdot, \cdot)\) chosen constant, the proofs of both lemma’s statements involve classical Lyapunov function techniques. More precisely, for the first-order case, consider the function defined by \(V_1(x) = x^2/2\) whose time-derivative along any solution to the closed-loop system is given by

\[
\dot{V}_1 = x \operatorname{sat}_m^M(-k_p x) \quad (\leq 0),
\]

with the time index omitted for the sake of notation simplification. The resulting boundedness of \(V_1(x)\) along any solution to the controlled system yields the stability of the point \(x = 0\), whereas the convergence of \(\dot{V}_1\) to zero, by application of Barbalat’s Lemma (see, e.g., Khalil (2002)) for instance, yields the convergence of \(x\) to zero, i.e. the desired convergence property.

As for the second-order system, assuming that \(a(\cdot, \cdot)\) is chosen positive constant, we note that the control (8) may be written as \(u = \operatorname{sat}_m^M(-(k_p x + g(\dot{x})))\), with

\[
g(s) = \frac{k_p |s| s}{2a} + \operatorname{sat}(k_v s).
\]

Therefore,

\[
g'(s) > \min \left\{ k_v, \frac{k_p l}{k_v a} \right\} > 0, \quad \forall s \in \mathbb{R},
\]

with \(g'(s)\) denoting the (right) derivative of \(g\) at \(s\). Moreover, \(\operatorname{sat}_m^M(s)s > 0, \forall s \neq 0\). Consider
the positive function
\[ V_2(x, \dot{x}) = \frac{k_p \dot{x}^2}{2} + \int_0^{k_p x + g(\dot{x})} \text{sat}_M(s) \, ds , \]
whose time-derivative along any solution to the closed-loop system is given by
\[ \dot{V}_2 = -g'(\dot{x})(\text{sat}_M(-(k_p x + g(\dot{x}))))^2 \leq 0 . \]

The resulting boundedness of \( V_2(x, \dot{x}) \) along any solution to the controlled system yields the stability of the point \((x, \dot{x}) = (0, 0)\). Since \( \dot{V}_2(t) \) is uniformly continuous, \( \dot{V}_2 \) tends to zero (Barbalat’s Lemma) and this convergence implies that \( \text{sat}_M(-(k_p x + g(\dot{x}))) \) tends to zero, and thus that \( k_p x + g(\dot{x}) \) tends to zero. Therefore, since \(|\dot{x}|\) is bounded, \( \frac{d}{dt}(x^2) + 2g(\dot{x})\dot{x}^2 \) tends to zero. Using the fact that \( g(\dot{x})/\dot{x} > 0 \), this in turn implies that \( x \) and \( \dot{x} \) tend to zero.

The proof of the lemma for the second-order system in the case where \( a(\cdot, \cdot) \) is the function defined by (7) is significantly more involved and, for this reason, is reported in Appendix A. \( \Box \)

Remark 1: The function \( a(\cdot, \cdot) \) defined by (7), which is proposed to make the control (8) an approximation of the TOC solution for any initial condition, is positive. It takes its values in the interval comprised between \(-m\) and \(M\). When \(|x|\) is not small, it is (almost) equal to \(-m\) if \(x\) is negative, and (almost) equal to \(M\) if \(x\) is positive. This means that in situations where the signs of possible initial values of \(x\) are the same (either all positive, or all negative), one can choose \(a(x, \varepsilon)\) constant and equal to either \(-m\) or \(M\), depending on the case, with no performance degradation. A situation of this type occurs with the longitudinal headway car control problem which is addressed in the last part of the paper.

We have seen so far that the controllers (4) and (8) are exponential stabilizers of the origins of the first-order and second-order integrator systems respectively, that they are continuous approximations of corresponding discontinuous TOCs, and that the approximations are all the better (in terms of functional approximation) than \(k_p\) is large and \(\varepsilon\) is small. As a matter of fact, these feedback controls are well-conditioned alternatives to the original TOCs when it comes to simulate the solutions to the controlled systems by using classical Runge-Kutta numerical integration packages. In practice, of course, the use of large control gains poses a number of well-known robustness problems in relation to various implementation issues (modeling errors, control discretization, noise measurement, etc.) so that the tuning of these gains is needed to reach an acceptable performance/robustness compromise. Nevertheless, an important practical shortcoming of these controllers is that they do not preserve the convergence to the desired equilibrium as soon as a non-zero constant—or slowly varying, in practice—additive perturbation acts on the system. Adding an integral action to the control law is the common way to correct this problem. In the next section, we propose a technique to complement the previously derived P and PD controllers with such an action by taking into account the bounds imposed on the control magnitude, with the concerns of limiting windup effects and of preserving the global asymptotic stability properties of the original controllers.

3 Integral action complementation

3.1 First-order system

We consider the first-order integrator with a complementary constant (unknown) perturbation input \(c\)
\[ \dot{x} = u + c . \]
We further assume that the perturbation magnitude is not too large. More precisely, we assume that \(|c| < \min\{M, -m\}\), with the latter inequality ensuring that the problem of global asymptotic stabilization of \(x = 0\) has a solution despite the bounds imposed on the control input. Rather than using a pure integrator of \(x\) in the control law, the authors of Seshagiri and Khalil (2005) propose to use a “bounded” integral term \(z\) calculated as follows

\[
\dot{z} = k_z(-z + \text{sat}^\delta_z(z + x)), \quad |z(0)| < \delta_z,
\]

with \(k_z\) and \(\delta_z\) denoting positive numbers. This relation indicates that we have a pure integrator \(\dot{z} = k_zx\) as long as \(|z + x| \leq \delta_z\), and also that \(|z(t)| \leq \delta_z\) and \(|\dot{z}(t)| \leq 2k_z\delta_z\), \(\forall t \geq 0\). Therefore, by defining \(\dot{z}_{\text{max}}\) as the maximal value that \(|\dot{z}(t)|\) is allowed to take, one has

\[
\delta_z = \frac{\dot{z}_{\text{max}}}{2k_z},
\]

and it is possible to modify the magnitudes of \(z\) and \(\dot{z}\) at will via the choice of the parameters \(\dot{z}_{\text{max}}\) and \(k_z\). We will see that the first of these parameters characterizes the importance given to the integral action at the control level, whereas \(k_z\) enters (in a simple way) the calculation of the gains associated with the linear PI controller of which the proposed nonlinear controller is a local approximation at \((x, z) = (0, 0)\). Let us proceed with the control design itself. A way to complement the nonlinear proportional feedback (4) with an integral action consists in conceptually replacing the initial state \(x\) by the modified state

\[
\bar{x} \equiv x + z,
\]

and determining a control which asymptotically stabilizes the augmented state \((\bar{x}, z) = (0, 0)\) when \(c \equiv 0\). In this case, and with the above definition of \(z\), the augmented control system writes as (using (9) with \(c = 0\), and (10))

\[
\begin{cases}
\dot{\bar{x}} = u + v(\bar{x}, z) \\
\dot{z} = v(\bar{x}, z)
\end{cases}
\]

with

\[
v(\bar{x}, z) \equiv k_z(-z + \text{sat}^\delta_z(z)).
\]

From here, the time-suboptimal feedback (4) can be used to asymptotically stabilize \(\bar{x} = 0\). Setting

\[
\begin{cases}
\dot{z}_{\text{max}} < \min\{M, -m\}, \\
M \equiv M - \dot{z}_{\text{max}} (> 0), \\
\bar{m} \equiv m + \dot{z}_{\text{max}} (< 0),
\end{cases}
\]

this yields the feedback controller

\[
u(\bar{x}, z) = \text{sat}_m^M(-k_p \bar{x}) - v(\bar{x}, z) = \text{sat}_m^M(-k_p \bar{x}) - k_z(-z + \text{sat}^\delta_z(\bar{x})),
\]

which, by construction, takes its values in the interval \([m, M]\), and whose linear approximation at \((\bar{x}, z) = (0, 0)\) is the linear PI controller \(u = -k_p x - k_i \int x\) with \(k_p = k_p + k_z\) and \(k_i = k_p k_z\). The corresponding closed-loop poles are real negative and equal to \(-k_p\) and \(-k_z\) respectively.
Lemma 3.1: The nonlinear PI feedback control (10)–(13) globally asymptotically stabilizes \((x, z) = (0, c/k_p)\) for the perturbed augmented system (9)–(10), provided that
\[
0 < \dot{z}_{\text{max}} < \min\{-m, M\} - |c|,
\]
\[
0 < k_z < \frac{k_p \dot{z}_{\text{max}}}{2\min\{-m, M\}}.
\]

Proof Define \(\bar{\bar{x}} \equiv \bar{x} - c/k_p\) and \(\bar{\bar{z}} \equiv z - c/k_p\). The desired stability property is equivalent to the global asymptotic stabilization of \((\bar{\bar{x}}, \bar{\bar{z}}) = (0, 0)\). From the system’s equation (9) and control expression, using the fact that \(\text{sat}_M(x - \varepsilon) + \varepsilon = \text{sat}_{M+c}(x)\), \(\forall x, \varepsilon,\)
one easily verifies that along any closed-loop solution
\[
\dot{\bar{\bar{x}}} = \dot{x} + \dot{z} = \text{sat}_m(-k_p \bar{x}) + c = \text{sat}_{M+c}(-k_p\bar{\bar{x}}).
\]
Using the condition upon \(\dot{z}_{\text{max}}\) one deduces that the time-derivative of \(\bar{\bar{x}}^2\) is negative whenever \(\bar{\bar{x}} \neq 0\). This in turn implies that \(\bar{\bar{x}}\) tends to zero and that \(\bar{x} = 0\) is globally asymptotically stable.

By noticing that \(\bar{x} = x + (z - c/k_p) = x + \bar{z}\), in order to show that the equilibrium \((x, z) = (0, c/k_p)\) of the augmented system (9)–(10) is asymptotically stable, it remains to show that \(\bar{z} = 0\) is asymptotically stable on the zero dynamics defined by \(\bar{x} = 0\). From (11), using the fact that the condition upon \(k_z\) implies that \(\delta_z > |c|/k_p\), and thus that \(\text{sat}^{\delta_z}(c/k_p) = c/k_p\), the evolution of \(\bar{z}\) on this zero dynamics is given by \(\dot{\bar{z}} = -k_z \bar{z}\). The desired property follows directly. \(\square\)

Remark 2: Choosing \(\dot{z}_{\text{max}}\) small does not impede the compensation of perturbations almost as large as the control bounds, and also limits the degradation of the control in terms of time-(sub)optimality. On the other hand, this imposes to use a small gain \(k_z\) with the risk of much penalizing the ultimate rate of convergence to the equilibrium. With these general rules in mind, the tuning of these parameters will essentially depend on the specific conditions and requirements of the application.

3.2 Second-order system

We now consider the second-order integrator with a constant (unknown) perturbation input \(c\)
\[
\ddot{x} = u + c.
\]
For the same reason as previously we assume that \(|c| < \min\{M, -m\}\). In the case of the first-order system, the relation (10) defines the way the bounded integral term \(z\) is calculated and ensures that the absolute value of \(\dot{z}\) is uniformly bounded by a chosen value. Concerning the second-order system, it is useful –for reasons that will clearly appear further in the paper– to ensure that the second-time derivative of \(z\) is uniformly bounded by a chosen value. We propose here to calculate \(z\) as follows
\[
\ddot{z} = -k_{vz} \dot{z} + \text{sat}^{\dot{z}_{\text{max}}/2}(k_{pz}(-z + \text{sat}^{\delta_z}(z + x))),
\]
with \(\dot{z}_{\text{max}}, \delta_z, k_{pz}, \text{and } k_{vz}\) denoting positive numbers, and with initial conditions such that \(|z(0)| < \delta_z + \dot{z}_{\text{max}}/(2k_{vz})\) and \(|\dot{z}(0)| < \dot{z}_{\text{max}}/(2k_{vz})\). One can verify (the proof is left as an exercise...
to the interested reader) that, whatever the evolution of $x(t)$, the absolute values of $z(t)$, $\dot{z}(t)$, and $\ddot{z}(t)$ are uniformly bounded by $\delta_z + \dot{z}_{\text{max}}/(2k_{uz})$, $\ddot{z}_{\text{max}}/(2k_{pz})$, and $\ddot{z}_{\text{max}}$ respectively. Away from saturation bounds, the evolution of $z$ is given by $\ddot{z} = -k_{uz}\dot{z} + k_{pz}x$ whose solutions can be approximated by those of the first-order equation $\dot{z} = (k_{pz}/k_{uz})x$ when $|\dot{x}|$ is small. This latter equation points out the integral action embedded in (15). Note also that, if $x + z = 0$ and $|z| < \ddot{z}_{\text{max}}$, then the evolution of $z$ is given by the autonomous second-order equation $\ddot{z} + k_{uz}\dot{z} + k_{pz}z = 0$. Therefore, on the zero dynamics defined by $x + z = 0$, the asymptotic exponential rate of convergence of $z$ to zero is proportional to $\omega = \sqrt{k_{pz}}$, whereas $\xi = k_{uz}/(2\sqrt{k_{pz}})$ is the damping factor (typically chosen between 0.7 and 1). Let us proceed with the control design by extending the method used for the first-order case. In view of (14) and (15), and defining again $\bar{x} \equiv x + z$, the augmented control system, in the case where $c \equiv 0$, now writes as

$$\begin{cases} \bar{x} = u + v(\bar{x}, z, \dot{z}) \\ \dot{z} = v(\bar{x}, z, \dot{z}) \end{cases}$$

with

$$v(\bar{x}, z, \dot{z}) \equiv -k_{uz}\dot{z} + \text{sat}^{\text{max}}_{\dot{z}}(k_{pz}(-z + \text{sat}^c(\bar{x}))).$$

Define

$$\begin{cases} \bar{M} \equiv M - \ddot{z}_{\text{max}} (> 0) \\ \bar{m} \equiv m + \ddot{z}_{\text{max}} (< 0) \end{cases}$$

The use of the time sub-optimal PD controller (8) in order to asymptotically stabilize $(\bar{x}, \dot{\bar{x}}) = (0,0)$, with $\bar{M}$ and $\bar{m}$ chosen as control bounds, yields the nonlinear PID feedback control

$$u(\bar{x}, \dot{\bar{x}}, z, \dot{z}) = \text{sat}^{\bar{M}}_{\bar{m}} \left(-k_p \left(\bar{x} + \frac{\dot{\bar{x}}|\dot{\bar{x}}|}{2a(\bar{x}, \varepsilon)}\right) - \text{sat}^c(k_{uz}\dot{z})\right) - v(\bar{x}, z, \dot{z}),$$

with $a(\bar{x}, \varepsilon)$ either positive constant or calculated according to (7), with $M$ and $m$ replaced by $\bar{M}$ and $\bar{m}$ respectively. Since $|v(\bar{x}, z, \dot{z})| < \ddot{z}_{\text{max}}$ along any solution to the controlled system, this control takes its values in the interval $[m, M]$. One easily verifies that the linear approximation of the (augmented) closed-loop system at $(\bar{x}, \dot{\bar{x}}, z, \dot{z}) = (0,0,0,0)$ is

$$\begin{cases} \ddot{x} = -k_p\bar{x} - k_{uz}\dot{z} \\ \ddot{z} = -k_{pz}z - k_{uz}\dot{z} + k_{pz}x \end{cases}$$

From these equations one can already deduce that the above controller is a (local) exponential stabilizer of $(\bar{x}, \dot{\bar{x}}, z, \dot{z}) = (0,0,0,0)$, and thus also of $(x, \dot{x}, z, \dot{z}) = (0,0,0,0)$, when $c \equiv 0$. The following lemma establishes a stronger asymptotic stability property when the perturbation $c$ is not exceedingly large (to the point of rendering the stabilization problem untractable).

**Lemma 3.2:** With $a(\bar{x}, \varepsilon)$ chosen either positive constant or according to relation (7) with $M$ and $m$ replaced by $\bar{M}$ and $\bar{m}$ respectively, the nonlinear PID feedback control (16)–(18) globally asymptotically stabilizes $(x, \dot{x}, z, \dot{z}) = (0,0,c/k_p,0)$ for the perturbed augmented system (14)–(15), provided that

$$\begin{cases} 0 < \ddot{z}_{\text{max}} < \min\{M, -m\} - |c| \\ \delta_z > \frac{|c|}{k_p} \\ 0 < \varepsilon < \frac{\min(-\bar{m}, \bar{M}) - |c|}{k_p} \end{cases}$$
Proof Let us first consider the case where \( a(\cdot, \cdot) \) is positive constant. Define \( \bar{x} \equiv \bar{x} - c/k_p \). One easily verifies that along any solution to the controlled system

\[
\ddot{\bar{x}} = \text{sat}_{M+c} \left( -k_p \left( \bar{x} + \frac{\dot{\bar{x}} |\bar{x}|}{2a} \right) - \text{sat}( -k_v \bar{x}) \right).
\]

From this equation and the condition imposed on \( \bar{z}_{\text{max}} \) one deduces, via a minor adaptation of the proof of Lemma 2.1, that \( \bar{z} \) is globally asymptotically stable. It then suffices to work on the zero dynamics defined by \( \bar{x} = 0 \), i.e. \( \bar{x} = c/k_p \), to prove the global asymptotic stability of \( (\bar{z}, \dot{\bar{z}}) = (c/k_p, 0) \). The technical arguments which justify the previous statement rigorously are classical and omitted for the sake of concision. Define \( \bar{z} = z - c/k_p \). In view of (15), when \( \bar{x} = c/k_p \) and \( \delta_z > |c|/k_p \), the evolution of \( \bar{z} \) is given by

\[
\ddot{\bar{z}} = -k_{uz} \bar{z} + \text{sat} \bar{z}^{\text{max}}/2 (-k_p \bar{z}) = -k_{uz} \bar{z} - h(\bar{z}) \bar{z},
\]
with \( h(\bar{z}) \equiv -(1/\bar{z}) \text{sat} \bar{z}^{\text{max}}/2 (-k_p \bar{z}) > 0, \forall \bar{z} \). Consider the positive function \( \mathcal{V} \) defined by

\[
\mathcal{V}(\bar{z}, \dot{\bar{z}}) = \frac{1}{2} \bar{z}^2 + \int_{s=0}^{\bar{z}} h(s) s \, ds.
\]

Using the above equation of evolution of \( \bar{z} \), the calculation of the time-derivative of this function yields \( \dot{\mathcal{V}} = -k_{uz} \bar{z}^2 < 0, \forall \bar{z} \neq 0 \). The stability of \( (\bar{z}, \dot{\bar{z}}) = (0, 0) \), which is equivalent to the stability of \( (z, \dot{z}) = (c/k_p, 0) \), is a direct consequence of the non-increasing of \( \mathcal{V}(\bar{z}, \dot{\bar{z}}) \). As for the convergence issue, \( \dot{\mathcal{V}} \) tends to zero by application of Barbalat’s Lemma, and so does \( \dot{\bar{z}} \). From there, one shows that \( \bar{z} \) is uniformly continuous and thus, by application of Barbalat’s Lemma, that \( \bar{z} \) tends to zero. In view of the equation of evolution of \( \bar{z} \), the convergence of \( \bar{z} \) and \( \bar{z} \) to zero in turn implies that \( \bar{z} \) tends to zero.

The proof for the more difficult case where \( a(\cdot, \cdot) \) is given by (7) is reported in Appendix B. It essentially involves minor adaptations of the proof of Lemma 2.1 reported in Appendix A. \( \square \)

Remark 3: A few words concerning the choice of the parameters \( \delta_z \) and \( \bar{z}_{\text{max}} \) are in order. The parameter \( \delta_z \) should be chosen larger than \( c/k_p \), as specified in the lemma, but not much larger in order to avoid the possible occurrence of uselessly large values of \( z \) favoring large overshoots. As for \( \bar{z}_{\text{max}} \), a compromise has to be found between a small value which minimizes the importance of the integral action, and thus also its negative effects (overshoot and performance degradation in terms of time optimality, in particular), and a larger value which allows for faster desaturation of the integral term \( z \).

4 Application to longitudinal headway car control

Alike other studies on this subject, the control design is here addressed by considering a simple model of the car’s dynamics with motorization and braking components schematized to the extreme, the idea being to work out a rough sketch of solutions before going to the stage of adaptation to an actual physical system. The problem statement and modeling equations here considered are basically those of Hatipoğlu et al. (1996), with the noticeable exception of aerodynamic and other drag forces which are not modeled in this reference. We indeed believe that it is important to take these forces into account from the beginning because the intensity of their sum rapidly increases approximately like the square of the vehicle’s velocity until its reaches it maximal value, corresponding to the vehicle’s maximal velocity, when it exactly matches the maximal traction force produced by the vehicle’s engine. Let us briefly recall the simplified longitudinal headway control problem that we are addressing:
the control variable $u$ is the vehicle’s acceleration/deceleration capacity. Assuming that the maximal motor-traction force $F_{motor}$ and braking force $F_{brake}$ are constant and known, together with the vehicle’s mass, one has $u \in [m, M]$ with $m = -\frac{F_{brake}}{mass}$ and $M = \frac{F_{motor}}{mass}$.

- the longitudinal dynamics of the controlled vehicle is given by Newton’s law

$$\ddot{d} = u - k_d |v| v,$$

with $d$ denoting the vehicle’s abscissa along the road, measured from an arbitrary fixed point, $v = \dot{d}$ the vehicle’s velocity, and $k_d$ the drag coefficient related to the vehicle’s maximal velocity $v_{max}$ by the relation

$$v_{max}^2 = \frac{M}{k_d}.$$

- the leading vehicle’s abscissa and velocity are denoted as $d_r$ and $v_r$ respectively, and the desired inter-distance between the two vehicles, here assumed constant and independent of $v_r$ for the sake of simplification, is denoted as $\Delta_r$.

Define $x = d - d_r + \Delta_r$, the control objective is to asymptotically stabilize $x = 0$ as efficiently as possible via the calculation of $u$. Combining the above model of the vehicle’s dynamics with the definition of $x$ yields the control system

$$\ddot{x} = u + c(\dot{x}, v_r, \dot{v}_r),$$

with

$$c(\dot{x}, v_r, \dot{v}_r) \equiv -k_d |\dot{x} + v_r| (\dot{x} + v_r) - v_r.$$

Although the perturbation $c$ is not constant in this case, it should tend to the constant value $-k_d |v_r| v_r$ when the leader’s velocity $v_r$ is constant. If $v_r$ varies slowly, it should also vary slowly. This intuitively justifies the idea of applying the nonlinear PID controller (16)-(18) of the previous section. The simulation results reported below have been performed by assuming that the inter-distance $d - d_r$ and the difference of velocities $v - v_r$ between the two vehicles are (precisely) measured on-line, so that $x$ and $\dot{x}$ are also measured and can be used directly in the control calculation. In practice, the measurement of the inter-distance can be obtained by using various optical devices (cameras, laser range-finders, etc.), but its time-derivative often has to be estimated. Such an estimator is proposed at the end of this section.

For the simulations, we have used the function $a$ given by (7) with $M$ and $m$ replaced by $\bar{M}$ and $\bar{m}$ respectively, and the various parameters involved in the system’s dynamics and control calculation have been chosen as follows

- Control bounds: $M = 3m/s^2$, $m = -9m/s^2$.
- Drag coefficient and maximal velocity: $k_d = 1.875 \times 10^{-3} m^{-1}$, $v_{max} = \sqrt{M/k_d} = 40 m/s$.
- Control gains and other parameters: $k_p = k_{pz} = 2$, $k_v = k_{vz} = 2 \sqrt{k_p}$, $\varepsilon = 1$, $l = 20$, $\ddot{z}_{max} = 0.1$, $\delta_z = (M - \ddot{z}_{max})/k_p = 1.45$. Larger control gains $k_p$ and $k_v$ would allow for a better approximation of the TOC solution, but would render the control more sensitive to measurement noise and data-acquisition/computation delays.

Moreover, knowing that the integral correction term $z$ is useful only when the error $x$ is not too large, we have modified the calculation of $z$, initially given by (15), as follows

$$\ddot{z} = -k_{vz} \dot{z} + sat_{\frac{v_{max}}{2}}(k_{pz}(-z + sat_{\frac{v_{max}}{2}}(z + x \ bell_{\nu}(x)))),$$
with

$$bell_{\nu,s}(x) = \frac{\tanh((x + \nu)/s) + \tanh((-x + \nu)/s)}{2\tanh(\nu/s)}$$

a bell-shaped symmetric function with a peak value equal to one at $x = 0$ and which tends to zero when $|x|$ tends to infinity. The parameter $\nu$ characterizes the width of the bell, i.e. the size of the interval in which the contribution of $x$ to the calculation and evolution of $z$ is the most important, whereas $s$ characterizes the steepness of the bell’s sides. For the reported simulations we have chosen $\nu = 10$ and $s = 1$. One easily verifies that this modification only changes the function $h$ in the proof of Lemma 3.2 without affecting its positivity property on which the result relies.

The results shown in the figures correspond to three situations involving three distinct leader’s velocities ($v_r = 0, 20, 35 \text{m/s}$). The initial inter-distance error is equal to 100 meters in all cases, and the follower’s velocity is initially equal to the leader’s velocity. The simulation figures show the time-evolution of i) the inter-distance (Fig. 1 and 2), ii) the difference of velocities between the follower and the leader (Fig. 3), iii) the control intensity (Fig. 4), and iv) the correction...
Figure 3. Velocity difference vs. time

Figure 4. Control vs. time

Figure 5. Integral term z vs. time
integral term (Fig. 5). They illustrate important performance differences between the proposed nonlinear PID controller and a (saturated) classical linear PID controller. In particular, the latter would yield important overshoots and several ineffective control-sign changes. One can also observe the consistency and robustness of the system’s response for different velocities that involve a large spectrum of drag forces, and the quasi absence of overshoot in all situations, despite the integral correction term which allows for the convergence of tracking error to zero. We also note that for large values of $|x|$ the sign of $x$ is always negative. This is coherent with the context of the application according to which the control is used to “catch up” with the leader, prior to stabilizing the inter-distance distance at zero. Therefore, as pointed out in a previous remark, the control can be implemented with $a(\cdot, \cdot)$ constant and equal to $-\bar{m}$ without a significant change in performance. However, in this latter case the control would not be optimal for (large) positive initial values of $x$ –a virtual possibility, since it means that the leader is initially (far) behind the follower.

For the sake of completeness, let us mention that an on-line estimation $\Delta_v$ of $\dot{x}$ based on the measurement of the inter-distance $x$ is obtained by considering the model $\ddot{x} = u + c$–with $c$ denoting a constant but unknown perturbation– for which a state estimator is given by

$$\begin{align*}
\dot{\Delta}_v &= k_1(x - \hat{x}) - k_2w + u \\
\dot{w} &= -k_3(x - \hat{x}) \\
\dot{\hat{x}} &= \Delta_v + k_4(x - \hat{x})
\end{align*}$$

with $k_i$ $(i = 1, \ldots, 4)$ denoting positive numbers. One easily verifies that the characteristic polynomial associated with the estimation-error model is $P(\lambda) = \lambda^3 + k_4\lambda^2 + k_1\lambda + k_2k_3$. Therefore, the transient performance of the estimator can be tuned via the choice of these coefficients, in relation to the poles of the linear approximation of the controlled system near the desired equilibrium. For instance, by setting $k_4 = 3\omega$, $k_1 = 3\omega^2$, and $k_2k_3 = \omega^3$, with $\omega > 0$, the three roots of the characteristic polynomial are real and equal to $-\omega$. Then, the larger $\omega$, the faster the estimation convergence rate, and the smaller the estimation error when the inter-distance velocity is not constant. However, in practice, the size of $\omega$ is limited by common high-gain sensitivity to noise/delay effects. Simulations that we have performed with $\omega = 2$ and $k_3 = 1$, and by using $\Delta_v$ instead of $\dot{x}$ in the control expression, did not show a significant difference with the case where $\dot{x}$ is directly measured.

5 Conclusion

Simple continuous and bounded nonlinear PI and PID controllers combining time-(sub)optimality with linear control robustness have been proposed for first-order and second-order integrator systems. Whereas most works on “proximate” time-optimal control assume that the control lower-bound and upper-bound are the opposite of each other, this restrictive assumption is not made here. A complementary contribution concerns the way of introducing an integral action with anti-windup properties into the control law, under the constraint of ensuring global asymptotic stability in the case of an additive constant (but unknown) perturbation acting on the system. The PID solution proposed for the second-order case has been applied to a (simplified) longitudinal headway car control problem, with unknown aerodynamic forces taken into account. Simulation results showing the performance of this solution have been reported.
Appendix A: Proof of Lemma 2.1, second-order case, when \( a(x, \varepsilon) \) is given by (7)

The closed-loop system equation is

\[
\ddot{x} = \text{sat}_m^M(-k_p x - g(\dot{x}, x)),
\]

(A1)

with

\[
g(\dot{x}, x) = \frac{k_p \dot{x} |\dot{x}|}{2a(x, \varepsilon)} + \text{sat}^l(k_v \dot{x}).
\]

First, one easily verifies the following facts:

F.1) \( \text{sign}(g(\dot{x}, x)) = \text{sign}(\dot{x}) \),

F.2) \( \Delta \equiv \min(-m, M) \leq a(x, \varepsilon) \leq \max(-m, M) \),

F.3) \( |x| > \varepsilon \Rightarrow \frac{\partial a}{\partial x}(x, \varepsilon) = \frac{\partial g}{\partial x}(\dot{x}, x) = 0 \).

Four technical propositions are proved next.

**Proposition A.1:** The solution \((x(t), \dot{x}(t))\) to the closed-loop system (A1) are uniformly bounded with respect to initial conditions and are uniformly continuous.

**Proof** Consider the following positive function

\[
U(x, \dot{x}) = -\frac{1}{k_p} \int_0^{k_p \dot{x}} \text{sat}_m^M(-s)ds + \frac{\dot{x}^2}{2},
\]

(A2)

whose time-derivative along any solution to (A1) is given by

\[
\dot{U} = -\dot{x}\text{sat}_m^M(-k_p x) + \dot{x}\text{sat}_m^M(-k_p x - g(\dot{x}, x)).
\]

(A3)

We show that \( \dot{U} \leq 0, \forall (x, \dot{x}) \) by considering the following four possible cases and by using F.1:

i) Case \( \dot{x} x = 0 \). Then \( \dot{U} = 0 \) if \( \dot{x} = 0 \), and \( \dot{U} = \dot{x}\text{sat}_m^M(-g(\dot{x}, x)) \leq 0 \) if \( x = 0 \).

ii) Case \( \dot{x} x > 0 \). Then \( |k_p x + g(\dot{x}, x)| \geq |k_p x| \) and \( \text{sign}(k_p x + g(\dot{x}, x)) = \text{sign}(\dot{x}) \). Therefore, \( \dot{x}\text{sat}_m^M(-k_p x - g(\dot{x}, x)) \leq \dot{x}\text{sat}_m^M(-k_p x) \), which is the same as \( \dot{U} \leq 0 \).

iii) Case \( \dot{x} x < 0 \) and \( \text{sign}(k_p x + g(\dot{x}, x)) \neq \text{sign}(x) \). One verifies that \( -\dot{x}\text{sat}_m^M(-k_p x) < 0 \) and \( \dot{x}\text{sat}_m^M(-k_p x - g(\dot{x}, x)) \leq 0 \) and, subsequently, that \( \dot{U} < 0 \).

iv) Case \( \dot{x} x < 0 \) and \( \text{sign}(k_p x + g(\dot{x}, x)) = \text{sign}(x) \). One verifies that \( -\dot{x}\text{sat}_m^M(-k_p x) < 0 \) and \( \dot{x}\text{sat}_m^M(-k_p x - g(\dot{x}, x)) > 0 \). One also verifies that \( |k_p x + g(\dot{x}, x)| < |k_p x| \) so that \( |\dot{x}\text{sat}_m^M(-k_p x - g(\dot{x}, x))| < |\dot{x}\text{sat}_m^M(-k_p x)| \). The negativity of \( \dot{U} \) follows from these inequalities.

The fact that \( \dot{U}(t) \) is non-increasing implies that \( x(t) \) and \( \dot{x}(t) \) are uniformly bounded with respect to initial conditions. The boundedness of \( \dot{x} \) in turn implies that \( x \) is uniformly continuous with respect to time. It also results from the boundedness of \( x \) and \( \dot{x} \) that \( \dot{x} \) is itself bounded, and thus that \( \dot{x} \) is uniformly continuous with respect to time \( \text{(end of proof of Proposition A.1)}. \)

**Remark A1:** Equation (A3) is not sufficient to prove the convergence of \( x(t) \) and \( \dot{x}(t) \) to zero, because the application of Barbilat’s lemma only ensures that \( \dot{U} \) tends to zero, whereas \( \dot{U} \) is equal to zero for all values of \( x \) and \( \dot{x} \) satisfying

\[
\min\{-k_p x, -k_p x - g(\dot{x}, x)\} \geq M, \quad \text{or} \quad \max\{-k_p x, -k_p x - g(\dot{x}, x)\} \leq m.
\]

This justifies the use of non-standard complementary arguments which are developed next.
**Proposition A.2:** If \( \dot{x} \) does not converge to zero, then for every pair of positive numbers \((t_0, \varepsilon)\), there exists a time-instant \( T > t_0 \) such that \(|x(T)| > \Delta/k_p - \varepsilon\).

**Proof** Let us prove Proposition A.2 by contradiction. Assume that there exists \( t_0 > 0 \) and \( \varepsilon > 0 \) such that \(|x(t)| \leq \Delta/k_p - \varepsilon, \forall t \geq t_0\). This supposition implies that

\[
\text{sat}_m^{-1}(-k_p x(t)) = -k_p x(t).
\]

When \( t \) is larger that \( t_0 \) one has

\[
\text{sat}_m^{-1}(-k_p x(t) - g(\dot{x}(t), x(t))) = -k_p x(t) - \lambda(t)g(\dot{x}(t), x(t))
\]

for some function \( \lambda \) such that

\[
0 < \frac{k_p \varepsilon}{\sup_{t \geq 0} g(\dot{x}(t), x(t))} \leq \lambda(t) \leq 1.
\]

From (A3) one then deduces that the time-derivative of \( \mathcal{U} \) defined by (A2) satisfies

\[
\dot{\mathcal{U}}(t) \leq -\lambda(t)\dot{x}(t)g(\dot{x}(t), x(t)) \leq -\lambda(t)\frac{k_p}{2\max(-m, M)}|\dot{x}(t)|^2 \leq 0, \quad \forall t \geq t_0.
\]

The convergence of \( \dot{\mathcal{U}}(t) \) to a limit value and the uniform continuity of \( \dot{\mathcal{U}}(t) \), which results from Proposition 1, implies that \( \dot{\mathcal{U}}(t) \) tends to zero by application of Barbalat’s lemma. The previous inequality in turn implies that \( \dot{x}(t) \) converges to zero. Proposition A.2 is a direct consequence of the resulting contradiction (end of proof of Proposition A.2).

**Proposition A.3:** For any positive number \( t_0 \), there exists a time-instant \( T > t_0 \) such that \(|x(T)| \leq \varepsilon\).

**Proof** Consider the positive function

\[
\mathcal{V}(x, \dot{x}) = \frac{1}{k_p} \int_0^t k_p x + g(\dot{x}) \text{sat}_m^M(s) \, ds + \frac{\dot{x}^2}{2},
\]

and consider a solution \((x(t), \dot{x}(t))\) to (A1). Assume that \(|x(t)| > \varepsilon\) for all \( t \) larger than some time instant \( t_0 \). Then, in view of F.3 one has

\[
\dot{\mathcal{V}}(t) = -\frac{\partial g}{\partial x}(\dot{x}(t), x(t))\left(\text{sat}_m^M(-k_p x(t) - g(\dot{x}(t), x(t)))\right)^2, \quad \forall t > t_0,
\]

with

\[
\frac{\partial g}{\partial x}(\dot{x}(t), x(t)) > \min \left\{ k_v, \frac{k_p l}{k_i \max(-m, M)} \right\} > 0, \quad \forall t > t_0.
\]

Therefore, \( \dot{\mathcal{V}}(t) \leq 0 \) and \( \mathcal{V}(t) \) converges to a limit value. By applying Barbalat’s lemma, one deduces that \( \dot{\mathcal{V}}(t) \) converges to zero. This in turn implies that \( k_p x(t) + g(\dot{x}(t), x(t)) \) converges to zero. Then, similarly to the proof of Lemma 2.1 when \( a(\cdot, \cdot) \) is constant, one deduces that \( x(t) \) and \( \dot{x}(t) \) also converge to zero. Proposition A.3 is a direct consequence of the resulting contradiction (end of proof of Proposition A.3).

The stability of \((x, \dot{x}) = (0, 0)\) results from the already proven boundedness of \( \mathcal{U} \) defined by (A2) with respect to initial conditions. It thus remains to prove that any solution \((x(t), \dot{x}(t))\) to
(A1) converges to $(0,0)$. Pick any positive number $\varepsilon$ smaller than $(\Delta/k_p - \varepsilon)/2$ (which is positive thanks to the condition on $\varepsilon$ in Lemma 2.1). Then

$$0 < \varepsilon < \frac{\Delta}{k_p} - 2\varepsilon < \frac{\Delta}{k_p} - \varepsilon.$$ 

Consider any solution to (A1). In the case where $\dot{x}$ does not converge to zero, Propositions A.2 and A.3 ensure the existence of a sequence of time interval $S_i \equiv [t_{i,1}, t_{i,2}]$, with $i \in \mathbb{N}$, such that (see Fig. A1)

- $t_{i,1} < t_{i,2} < t_{i+1,1}$,
- $x(t_{i,1}) = \pm \varepsilon$,
- $x(t_{i,2}) = \pm (\Delta/k_p - \varepsilon)$,
- $\forall t \in S_i : |x(t)| \leq \Delta/k_p - \varepsilon$.

![Time-intervals $S_i \equiv [t_{i,1}, t_{i,2}]$ when $\dot{x}$ does not converge to zero.](image)

Note that the existence of this sequence and the fact that $\dot{x}$ is bounded also implies that the difference $(t_{i,2} - t_{i,1})$ is larger than some positive number, and that $t_{i,1}$ tends to infinity with $i$.

**Proposition A.4:** If $\dot{x}$ does not converge to zero, then there exists $\varepsilon_v > 0$ such that $|\dot{x}(t)| \geq \varepsilon_v$, $\forall i \in \mathbb{N}$, $\forall t \in S_i$.

**Proof** Assume that there exists $i \in \mathbb{N}$ and $t_{i,3} \in S_i = [t_{i,1}, t_{i,2}]$ such that $|\dot{x}(t_{i,3})| < \varepsilon_v := \sqrt{\varepsilon(2\Delta - k_p \varepsilon)}$, then

$$U(t_{i,3}) = -\frac{1}{k_p} \int_0^{k_p x(t_{i,3})} \text{sat}_m^M(-s) ds + \frac{\dot{x}(t_{i,3})^2}{2}$$

$$= \frac{1}{k_p} \int_0^{k_p x(t_{i,3})} sds + \frac{\dot{x}(t_{i,3})^2}{2}$$

$$< \frac{k_p}{2} \left( \frac{\Delta}{k_p} - \varepsilon \right)^2 + \frac{\varepsilon(2\Delta - k_p \varepsilon)}{2} = \frac{\Delta^2}{2k_p}$$

$$< -\frac{1}{k_p} \int_0^\Delta \text{sat}_m^M(-s) ds.$$ 

This strict inequality implies the existence of a constant $\bar{\varepsilon} > 0$ such that

$$U(t_{i,3}) \leq -\frac{1}{k_p} \int_0^\Delta -k_p \varepsilon \text{sat}_m^M(-s) ds.$$
Since $U(t)$ is non-increasing,

$$-\frac{1}{k_p} \int_0^{k_p x(t)} \text{sat}_m(-s)\,ds \leq U(t) \leq U(t_{i,3}) \leq -\frac{1}{k_p} \int_0^{\Delta - k_p \bar{e}} \text{sat}_m(-s)\,ds,$$

when $t$ is larger than $t_{i,3}$. Therefore, $|x(t)| < \Delta/k_p - \bar{e}$, $\forall t \geq t_{i,3}$. The resulting contradiction with Proposition A.2 proves Proposition A.4 \textit{(end of proof of Proposition A.4)}. 

Now, from Proposition A.4, if $\dot{x}$ does not converge to zero, one deduces, in view of (A3) that

$$\dot{U}(t) = -\dot{x}(t) \text{sat}_m(-k_p x(t)) + \dot{x}(t) \text{sat}_m(-k_p x(t) - g(\dot{x}(t), x(t)))$$

$$= k_p \dot{x}(t)x(t) + \dot{x}(t) \left(-k_p x(t) + \text{sat}_m^{M+k_p x(t)}(-g(\dot{x}(t), x(t)))\right)$$

$$= \dot{x}(t) \text{sat}_m^{M+k_p x(t)}(-g(\dot{x}(t), x(t))) \quad \text{(A4)}$$

for all $t \in S_i$. One also easily verifies that, for all $t \in S_i$,

$$m + k_p x(t) \leq -\Delta + k_p x(t) < -k_p \bar{e} < k_p \bar{e} < \Delta + k_p x(t) < M + k_p x(t).$$

From (A4) and Fact F.1 one deduces the existence of a positive constant $\gamma$ such that, for all $t \in S_i$,

$$\dot{U}(t) \leq -\dot{x}(t) \text{sat}_m^{k_p \bar{e}}(g(\dot{x}(t), x(t))) \leq -\gamma \varepsilon^2_v. \quad \text{(A5)}$$

Moreover, as already pointed out, the difference $(t_{i,2} - t_{i,1})$ is larger than some positive constant $\delta$, due to the boundedness of $|\dot{x}|$. Therefore, using (A5) and the fact that $U$ is non-increasing, one deduces that

$$U(+\infty) = U(0) + \int_0^{+\infty} \dot{U}(t)\,dt$$

$$\leq U(0) + \sum_{i=1}^{+\infty} \int_{t_{i,1}}^{t_{i,2}} \dot{U}(t)\,dt$$

$$\leq U(0) - \sum_{i=0}^{+\infty} \delta \gamma \varepsilon^2_v = -\infty,$$

which contradicts the fact that $U$ is a positive function.

This contradiction in turn implies that the initial assumption according to which $\dot{x}(t)$ does not converge to zero is false. Therefore, $\dot{x}(t)$ tends to zero. By Barbalat’s lemma, $\ddot{x}(t)$ also tends to zero, since $\ddot{x}(t)$ is uniformly continuous in view of (A1) and the uniform continuity of $x(t)$ and $\dot{x}(t)$. Finally, (A1) also shows that the convergence of $\dot{x}(t)$ and $\ddot{x}(t)$ to zero is possible only if $x(t)$ itself tends to zero.
Appendix B: Sketch of proof of Lemma 3.2 when \( a(\bar{x}, \varepsilon) \) is given by (7)

One verifies that along any solution to the controlled system one has

\[
\ddot{x} = \text{sat}_2^{\bar{M} + c} \left( -k_p \dot{x} - g(\dot{x}, \ddot{x}) \right),
\]

with

\[
g(\dot{x}, \ddot{x}) = \frac{k_p |\dot{x}|}{2a(\dot{x}, \varepsilon)} + \text{sat}^l(-k_v \dot{x}),
\]

\[
a(\bar{x}, \varepsilon) = \frac{\bar{M} - \bar{m}}{2} - \frac{\bar{M} + \bar{m}}{2} \text{sat}^l \left( \frac{\bar{x} + c/k_p}{\varepsilon} \right).
\]

From there, modulo minor adaptations, the proof proceeds similarly to the proof in Appendix A in order to show that \( \bar{x}(t) \) and \( \dot{\bar{x}}(t) \) tend to zero. For instance, the facts F.1–F.3 must be replaced by

F.1) \( \text{sign}(g(\dot{x}, \ddot{x})) = \text{sign}(\dot{x}) \),
F.2) \( \Delta \equiv \min(-\bar{m}, \bar{M}) \leq a(\bar{x}, \varepsilon) \leq \max(-\bar{m}, \bar{M}) \),
F.3) If \( \bar{x} > \varepsilon - c/k_p \) or \( \bar{x} < -\varepsilon - c/k_p \), then \( \frac{\partial a}{\partial g}(\bar{x}, \varepsilon) = \frac{\partial g}{\partial g}(\dot{x}, \ddot{x}) = 0 \).

Propositions A.1 and A.2 still hold with \( x \) and \( \dot{x} \) replaced by \( \bar{x} \) and \( \dot{\bar{x}} \) respectively, and system (A1) replaced by (B1). Note that the positive function \( U \) defined by (A2) is used for the proof of these propositions, where \( x \) and \( \dot{x} \) are replaced by \( \bar{x} \) and \( \dot{\bar{x}} \) respectively, and \( m \) and \( M \) are replaced by \( \bar{m} \) and \( \bar{M} \) respectively. Also, following the adaptation of Fact F.3, Proposition A.3 becomes:

\[
\forall t_0, \exists t > t_0 \text{ such that } -\varepsilon - c/k_p \leq \bar{x}(t) \leq \varepsilon - c/k_p.
\]

Then, pick any positive number \( e \) smaller than \( \frac{1}{2} \left( \frac{\Delta - |c|}{k_p} - \varepsilon \right) \). Note that the conditions on \( \bar{z}_{\max} \) and \( \varepsilon \) in the lemma’s statement ensure the existence of such a number \( e \). Let us assume that \( \dot{\bar{x}} \) does not converge to zero. Then, Propositions A.2 and A.3 ensure the existence of a sequence of time interval \( S_i \equiv [t_{i,1}, t_{i,2}] \), with \( i \in \mathbb{N} \), such that

\bullet \quad t_{i,1} < t_{i,2} < t_{i+1,1},
\bullet \quad x(t_{i,1}) \text{ equal to either } \varepsilon - c/k_p \text{ or } -\varepsilon - c/k_p,
\bullet \quad x(t_{i,2}) = \pm(\Delta/k_p - e),
\bullet \quad \forall t \in S_i : |x(t)| \leq \Delta/k_p - e.

The condition upon the choice of \( e \) ensures that

\[
-\Delta/k_p + e < -\varepsilon - c/k_p < \varepsilon - c/k_p < \Delta/k_p - e,
\]

which is useful to validate the fourth property of the sequence \( \{S_i\} \). Proposition A.4 is then obtained with \( \dot{x} \) replaced by \( \dot{\bar{x}} \). From there, one proceeds exactly as in the proof in Appendix A to show a contradiction with the assumption that \( \bar{x} \) does not converge to zero. Global asymptotic stability of \( (\bar{x}, \dot{\bar{x}}) = (0,0) \) follows. Finally, global asymptotic stability \( (x, \dot{x}, z, \dot{z}) = (0,0,c/k_p,0) \) is obtained as in the proof of Lemma 3.2 when \( a(\cdot, \cdot) \) is constant.
References


